Critical Mode Interaction in the Presence of External Random Excitation

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The effects of random external excitation on nonlinear systems with marginally stable/unstable modes is studied within the context of stochastic bifurcation theory. Using the method of stochastic averaging, a Markov approximation is derived and a perturbation technique is developed to solve the resulting Fokker-Planck equation. It is found that, due to mode interaction through system nonlinearities, deterministic bifurcation characteristics are not preserved in the presence of external random excitation. In general, one critical mode can experience a stabilizing effect at the expense of the other. The theory is then applied to a flight dynamics problem at large angles of attack and sideslip.

Nomenclature‡

A, A_c, A_s = square matrices, dimension n, n_c, and n_s, respectively
C_{mn} = coefficient of trial $r^m z^n$
$d_c(t)$, $d_s(t)$ = nonwhite external random excitation
E = expectation operator
$f(x)$, $f_c(x)$, $f_s(x)$ = nonlinear vector fields
f_{klm}, g_{klm}, h_{klm} = coefficient of the nonlinear term: $x^k y^l z^m$
G = $n \times m$ constant matrix
J = probability flux
L_0, L_1 = perturbation operators of the Fokker-Planck equation
$M(m,n)$ = the moment $E[r^m z^n]$
m_0, m_1 = $n \times 1$ drift vectors
n = total number of state variables, $= n_c + n_s$
n_c, n_s = number of critical and stable state variables, respectively
$P_g(t)$ = external excitation of the roll rate
p = roll rate
Q, $Q1$ = $n \times n$ diffusion matrices
R = correlation matrix
r = amplitude (polar coordinate), also yaw rate
S, S_1, S_2 = excitation intensities
$W(t)$ = vector of Wiener processes
x = n vector of state variables, $= \{x_c, x_s\}^T$
x_c, x_s = vectors of critical and stable variables, respectively
x,y,z = Cartesian coordinates
β = sideslip angle
δ = deviation from potential flow conditions
ϵ = scaling parameter
$\eta(t)$ = m vector of white noise
θ = phase angle (polar coordinate)
μ = vector of system parameters
μ,λ = unfolding parameters
μ_c, λ_c = bifurcation points
$\Phi(\omega)$ = power spectral density
ϕ = roll angle
ω = frequency, rad/s

Subscripts
i = ith component
ij = ijth element of a matrix

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‡The summation convention denoted by repeated subscripts is used.

Introduction

THE conventional linear model for the flight dynamics of rigid aircraft is valid for small angles of attack and sideslip. At large angles of attack and sideslip, the aerodynamic stability derivatives are no longer constant and, hence, a nonlinear analysis is required. Rhoads and Schuler[1] were one of the first to perform a theoretical and experimental study of airplane dynamics in large-disturbance maneuvers. A key feature of their work is the dependence of the stability derivatives on the Mach number and angle of attack. The effect of large sideslip angles was considered by Ross.[2] Based on both flight test and wind-tunnel test results, a cubic dependence of the yawing moment on the sideslip angle was proposed, and it was demonstrated that an unstable Dutch roll mode gave rise to an oscillation of the aircraft wing about the roll axis. This phenomenon is commonly referred to as "wing rock." The work was followed by investigations of cubic nonlinearities in the rolling moment as well as the damping-in-roll derivatives.[3] In this case, both directional divergence as well as wing rock was accounted for.

Various techniques are available for the analysis of nonlinear systems. One such method is the pseudosteady-state analysis of Young et al.[4] Neglecting the effects of gravity, a fifth-order model was derived. Equilibrium solutions of such a system were referred to as pseudosteady. Another approach is the use of bifurcation theory by Carroll and Mehra[5] and Hui and Tobak,[6] where the full six-DOF model was employed in conjunction with a nonlinear aerodynamics model based on wind-tunnel tests. In the context of bifurcation theory, the Dutch roll/wing-rock instability observed by Ross corresponds to a supercritical Hopf bifurcation. This phenomenon is characterized by a pair of complex eigenvalues crossing the imaginary axes. The loss of directional stability is characterized by the crossing of a real eigenvalue. This is referred to as a simple bifurcation. The dymanics of systems with eigenvalues close to the imaginary axes (i.e., marginally stable/unstable or critical modes) was studied by Cochran and Ho[7] using Malkin's method. Malkin's method is related to the theory of center manifold. Basically, the theory of center manifold is a technique whereby the marginally stable/unstable modes are decoupled from the stable modes and the nonlinear analysis can then be performed on a subsystem of lower dimension.

The extension of these techniques for the analysis of nonlinear systems perturbed by random excitation has been developed by Haken,[8] Horsthemke and Lefever,[9] Papanicoloau and Kohler,[10] Knobloch and Wisenfeld,[11] Coullet et al.,[12] and Namachchivaya and Leng.[13] In this paper, the effects of external random excitation on systems with marginally stable/unstable modes will be examined. Physically, this corresponds to flight at large angles of attack and sideslip in a turbulent atmosphere. In particular, the question of whether bifurcation characteristics of the deterministic nonlinear model are preserved in the presence of random external excitation is posed. The requisite stochastic bifurcation theory is stated in the next section. This will be followed by examples of systems with one or two critical modes. Finally, an application to a lateral flight dynamics problem will be demonstrated.

Problem Formulation

The model for the flight dynamics of rigid aircraft takes the form of a system of first-order ordinary nonlinear differential equations denoted by

$$x' = f(x,\mu) \tag{1}$$

The equilibrium or fixed points of the system are obtained by solving $f(x,\mu) = 0$. Sufficiently close to a fixed point, one may approximate the dynamics with a linear model defined by

$$x' = A(\mu)x \tag{2}$$

where the Jacobian $A = [\partial f/\partial x]$ is evaluated at the fixed point and is assumed to be nonsingular there. The stability of the fixed point is then defined by the eigenvalues of the Jacobian matrix. These eigenvalues λ are given by the roots of the characteristic equation:

$$\det[A(\mu) - \lambda I] = 0 \tag{3}$$

which reduces to an nth-deg polynomial. For the fixed point to be asymptotically stable, these eigenvalues must have negative real parts. Changes in the qualitative nature or number of fixed points as the system parameters μ are varied are called bifurcations. The values of the system parameters μ_c at which these changes occur are called bifurcation points. Thus, bifurcation points are the values of the system parameters at which eigenvalues of the linearized system cross the imaginary axis. In practice, instability often occurs with the crossing of one or at most two eigenvalues. Thus, it is convenient to partition Eq. (1) into critical x_c and stable x_s modes with the dimension of the stable subsystem usually much larger then the dimension of the critical subsystem. This is denoted by

$$\begin{Bmatrix} x_c' \\ x_s' \end{Bmatrix} = \begin{bmatrix} A_c & 0 \\ 0 & A_s \end{bmatrix} \begin{Bmatrix} x_c \\ x_s \end{Bmatrix} + \epsilon^2 \begin{Bmatrix} f_c(x_c,x_s) \\ f_s(x_c,x_s) \end{Bmatrix} + \epsilon \begin{Bmatrix} d_c(t) \\ d_s(t) \end{Bmatrix} \tag{4}$$

Without loss of generality, A_c and A_s are assumed to be in canonical form. The eigenvalues of A_c have near zero real parts, and the eigenvalues of A_s have negative real parts. The system's nonlinearities, typically polynomials in the state variables x_c and x_s, are represented by $f_c(x_c,x_s)$ and $f_s(x_c,x_s)$. In flight dynamics, these nonlinear terms are usually obtained by interpolation of flight tests and wind-tunnel test results. The parameter ϵ is used to scale the interaction between the deterministic system and the external excitation $d_c(t)$ and $d_s(t)$.

The crossing of a real eigenvalue corresponds to a loss of static stability, and in the lateral dynamics of rigid aircraft, this is associated with the spiral mode. For this case, $A_c = \epsilon^2\mu$ and the bifurcation point is simply $\mu_c = 0$. The parameter μ, which measures the marginal stability/instability of the system, is referred to as an unfolding parameter in bifurcation theory; the number of unfolding parameters present is called the codimension of the bifurcation. The crossing of a complex pair of eigenvalues causes an oscillatory or dynamic instability (e.g., Dutch roll mode). For this case, A_c is given as

$$A_c = \begin{bmatrix} \epsilon^2\mu & -\omega \\ \omega & \epsilon^2\mu \end{bmatrix}$$

and is also classified as a codimension one bifurcation even though the critical subsystem is two dimensional. For systems with marginally stable/unstable modes, the nature of the bifurcation or loss of stability, as system parameters are varied, is characterized by certain combinations of the nonlinear terms present. In bifurcation theory, these special combinations are called normal forms. The normal forms for the instabilities mentioned earlier and their relation to the actual system nonlinearities may be found in Guckenheimer and Holmes.[14] The least undesirable loss of stability is a soft loss of stability whereby the transition to a new equilibrium takes place gradually as system parameters are varied. Henceforth, particular attention will be paid to this mode of instability.

Now consider the system perturbed by external white noise, where the dependence of f on the system parameters μ is implied:

$$x' = f(x) + G\eta(t) \tag{5}$$

This system may be represented as an Itô stochastic differential equation[15]:

$$\mathrm{d}x = f(x)\mathrm{d}t + G\,\mathrm{d}W(t) \tag{6}$$

The probability density function of the response x, governed by the Ito system [Eq. (6)], is then given by a parabolic partial differential equation called the Fokker-Planck equation (FPE):

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x_i}\left[-f_i(x)p + \frac{1}{2}\frac{\partial(Q_{ij}p)}{\partial x_j}\right] \tag{7}$$

where Q is defined by

$$Q = G\,E[\eta(t)\eta^T(t)]\,G^T \tag{8}$$

The contribution from the deterministic system $f_i(x)$ is referred to as the drift component, whereas the term Q_{ij} is called the diffusion component. In the event that the external random excitation is nonwhite but sufficiently broadband, a white noise approximation for the critical modes can be made by means of the extended stochastic averaging theorem of Papanicoloau and Kohler.[10] The important consequence of this theorem is that, insofar as the bifurcation behavior is concerned, the stable modes may be ignored for the case of external random excitation. Hence, there is no need to retain the full six-DOF model. Furthermore, it has been shown[13] that the drift term of the white noise approximation is basically the normal form of the deterministic system. Therefore, from now on, one need only consider the effect of random excitation on the deterministic normal form.

Systems with One Critical Mode

Exact solutions to the steady-state FPE can be found easily for one-dimensional systems. The FPE takes the form of a continuity equation:

$$\frac{\partial p}{\partial t} = \frac{\partial J}{\partial x_i} \tag{9}$$

where the probability mass flow (flux) J is defined as

$$J = -f_i(x)p + \frac{1}{2}\frac{\partial(Q_{ij}p)}{\partial x_j} \tag{10}$$

For one-dimensional systems, a steady state ($\partial p/\partial t = 0$) exists if the probability mass flow is zero everywhere. This condition is referred to as the potential flow condition.[16] Now consider a system with one critical mode undergoing a soft loss of stability (pitchfork bifurcation):

$$x' = \epsilon^2(\mu x + C_3x^3) + \epsilon\eta(t), \qquad (C_3<0) \tag{11}$$

The white noise approximation represented as an Itô system is

$$dx = \epsilon^2(\mu x + C_3x^3)dt + \epsilon S\,dW(t) \tag{12}$$

where $S^2 = E[d(t+\tau)d(t)]$ and solving the corresponding FPE yields

$$p_s(x) = N\exp[2(\mu x^2/2 + C_3x^4/4)/S^2] \tag{13}$$

where the normalizing constant N is given by

$$N^{-1} = \int_{-\infty}^{\infty} \exp[2(\mu x^2/2 + C_3x^4/4)/S^2]\,dx \tag{14}$$

The extrema of the probability density function (pdf) are defined by $dp_s(x)/dx = 0$, and these extrema correspond to the fixed points of the deterministic system. Furthermore, changes in the number of extrema and their nature occur at the same bifurcation points as the deterministic system (i.e., $\mu_c = 0$). Hence, bifurcation characteristics of the deterministic system are preserved in the presence of external random excitation.

For the Hopf bifurcation, a similar analysis may be performed. Consider the normal form for a soft transition ($a<0$):

$$x' = \epsilon^2\mu x - \omega y + \epsilon^2(ax+by)(x^2+y^2) + \epsilon\eta_1(t) \tag{15a}$$

$$y' = \omega x + \epsilon^2\mu y + \epsilon^2(ay-bx)(x^2+y^2) + \epsilon\eta_2(t) \tag{15b}$$

Applying the extended stochastic theorem, the white noise approximation is represented in polar coordinates by the Itô system:

$$dr = \epsilon^2[\mu r + ar^3 + (\sigma^2/2r)]dt + \epsilon S\,dW_1(t) \tag{16a}$$

$$d\theta = (\omega + \epsilon^2br^2)dt + \epsilon S/r\,dW_2(t) \tag{16b}$$

$W_1(t)$ and $W_2(t)$ are independent Wiener processes and S is given by

$$S^2 = \frac{1}{2}\int_{-\infty}^{\infty} \{\cos\omega\tau[R_{11}(\tau) + R_{22}(\tau)] + \sin\omega\tau[R_{12}(\tau) - R_{21}(\tau)]\}d\tau \tag{17a}$$

where

$$R_{ij}(\tau) = E[\eta_i(t+\tau)\eta_j(t)] \tag{17b}$$

The steady-state FPE is then explicitly:

$$\begin{aligned} 0 &= -\frac{\partial}{\partial r}\left[\left(\mu r + ar^3 + \frac{\sigma^2}{2r}\right)p_s(r,\theta) - \frac{\sigma^2\partial p_s}{2\partial r}\right] \\ &= -\frac{\partial}{\partial\theta}\left[(\omega + \epsilon^2br^2)p_s(r,\theta) - \frac{\sigma^2\partial p_s}{2r^2\partial\theta}\right] \end{aligned} \tag{18}$$

and the solution is

$$\begin{aligned} p_s(r,\theta) &= p_s(\theta)p_s(r) \\ &= \frac{1}{2\pi}\cdot Nr\exp\left[\frac{2}{\sigma^2}\left(\frac{\mu}{2}r^2 + \frac{a}{4}r^4\right)\right] \end{aligned} \tag{19}$$

Bringing the steady-state pdf back to Cartesian coordinates by means of the relation

$$p_s(x,y)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = p_s(r,\theta)$$

with $x = r\cos\theta$ and $y = r\sin\theta$ and $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$, yields

$$p_s(x,y) = N'\exp\left\{\frac{2}{\sigma^2}\left[\frac{\mu}{2}(x^2+y^2) + \frac{a}{4}(x^2+y^2)^2\right]\right\} \tag{20}$$

The extrema of $p_s(x,y)$ are determined by

$$\begin{aligned} 0 &= \frac{\partial p_s}{\partial x} = \frac{2}{\sigma^2}[\mu x + a(x^2+y^2)x]p_s(x,y) \\ 0 &= \frac{\partial p_s}{\partial y} = \frac{2}{\sigma^2}[\mu y + a(x^2+y^2)y]p_s(x,y) \end{aligned} \tag{21}$$

which leads to $x = y = 0$ and $(x^2+y^2) = -\mu/a$ (for $\mu>0$), which coincide with the fixed point/limit cycle of the deterministic [$\boldsymbol{\eta}(t) = \mathbf{0}$] normal form [Eqs. (15)]. Qualitative changes in the nature of the pdf also occur at the same bifurcation points as the deterministic system (i.e., $\mu_c = 0$). Hence, the bifurcation characteristics of a Dutch roll/wing-rock instability are robust to external excitation.

Systems with Two Critical Modes

For the case of an aircraft with marginally stable/unstable Dutch roll and spiral modes, the relevant quantities, in the notation of Eq. (4), are explicitly

$$x_c = \{x,y,z\} \tag{22a}$$

$$A_c = \begin{bmatrix} \epsilon^2\mu & -\omega & 0 \\ \omega & \epsilon^2\mu & 0 \\ 0 & 0 & \epsilon^2\lambda \end{bmatrix} \tag{22b}$$

$$f_c(x,y,z,\mathbf{0}) = \{f_{klm}x^ky^lz^m,\ g_{klm}x^ky^lz^m,\ h_{klm}x^ky^lz^m\}^T$$

$$k+l+m=3 \tag{22c}$$

$$d_c(t) = \{d_1(t),\ d_2(t),\ d_3(t)\}^T \tag{22d}$$

where x,y represents the Dutch roll mode and z the spiral divergence mode. Nonlinearities involving the stable state variables may be ignored and cubic nonlinearities in the critical state variables are considered for a soft loss of stability. The normal form for the system is given by

$$x' = \epsilon^2\mu x - \omega y + \epsilon^2[(cx-jy)(x^2+y^2) + (ex-hy)z^2] + \epsilon\,\eta_1(t) \tag{23a}$$

$$y' = \omega x + \epsilon^2\mu y + \epsilon^2[(jx+cy)(x^2+y^2) + (ey+hx)z^2] + \epsilon\,\eta_2(t) \tag{23b}$$

$$z' = \epsilon^2[\lambda z + d(x^2+y^2)z + bz^3] + \epsilon\,\eta_3(t) \tag{23c}$$

where the parameters b,c,d,e,g,h, after averaging, are related to the system nonlinearities as follows:

$$b = h_{003}, \qquad c = (3f_{300} + f_{120} + g_{210} + 3g_{003})/8 \tag{24a}$$

$$d = (h_{201} + h_{021})/2, \qquad e = (f_{102} + g_{012})/2 \tag{24b}$$

$$j = (-3f_{030} - f_{210} + g_{120} + 3g_{300})/8,$$

$$h = (-f_{012} + g_{102})/2 \tag{24c}$$

Making the white noise approximation, the Itô system is

$$dr = \epsilon^2(\mu r + erz^2 + cr^3 + (S_1)^2/2r)dt + \epsilon S_1\, dW_1(t) \tag{25a}$$

$$d\theta = [\omega + \epsilon^2(jr^2 + hz^2)]dt + \epsilon S_1/r\, dW_2(t) \tag{25b}$$

$$dz = \epsilon^2(\lambda z + dr^2 z + bz^3)dt + \epsilon S_2\, dW_3(t) \tag{25c}$$

where S_1 and S_2 are defined by

$$S_1^2 = \frac{1}{2}\int_{-\infty}^{\infty} \{\cos(\omega\tau)[R_{11}(\tau) + R_{22}(\tau)] + \sin(\omega\tau)[R_{12}(\tau) + R_{21}(\tau)]\}d\tau \tag{26a}$$

$$S_2^2 = \int_{-\infty}^{\infty} R_{33}(\tau)d\tau \tag{26b}$$

$$R_{ij}(\tau) = E[\eta_i(t+\tau)\eta_j(t)] \tag{26c}$$

The steady-state FPE takes the form:

$$0 = -\frac{\partial J_r}{\partial r} - \frac{\partial J_\theta}{\partial \theta} - \frac{\partial J_z}{dz}$$

with

$$J_r = \left[\mu r + erz^2 + cr^3 + \frac{(S_1)^2}{2r}\right]p_s - \frac{(S_1)^2}{2}\frac{\partial p_s}{\partial r} \tag{27a}$$

$$J_\theta = [\omega + \epsilon^2(jr^2 + hz^2)]p_s - \frac{(S_2)^2}{2r^2}\frac{\partial p_s}{\partial r} \tag{27b}$$

$$J_z = [\lambda z dr^2 z + bz^3]p_s - \frac{(S_2)^2}{2}\frac{\partial p_s}{\partial z} \tag{27c}$$

The present situation is not unlike that of the Hopf bifurcation considered earlier. Let $p_s(r,z,\theta)$ take the form:

$$p_s(r,z,\theta) = p_s(\theta)p_s(r,z) = (1/2\pi)\, p_s(r,z) \tag{28}$$

The θ component of the FPE $(\partial J_\theta/\partial\theta)$ is then identically zero. This reduces the dimension of the FPE by one. It is convenient to rescale the variables as follows:

defining the new state variables

$$r' = \sqrt{2}\, r/S_1, \qquad z' = \sqrt{2}\, z/S_2 \tag{29}$$

defining the new system parameters

$$b' = bS_2^2/2, \quad c' = cS_1^2/2, \quad d' = dS_1^2/2$$

$$e' = eS_2^2/2 \tag{30}$$

The FPE may now be written as

$$0 = -\frac{\partial J_r}{\partial r} - \frac{\partial J_z}{\partial z}$$

with

$$J_r = \mu r + (k-\delta)rz^2 + cr^3 + 1/r - \frac{\partial p}{\partial r} \tag{31a}$$

$$J_z = \lambda z + (k+\delta)r^2 z + bz^3 - \frac{\partial p}{\partial z} \tag{31b}$$

with the understanding that all variables and parameters have been rescaled (primes are omitted), and k and δ, for reasons that will be apparent subsequently, are defined by

$$\delta = (d' - e')/2, \qquad k = (d' + e')/2 \tag{32}$$

A potential flow solution defined by J_r and $J_z = 0$ at the steady state can be obtained for the case $\delta = 0$, as

$$p_s(r,z) = Nr\exp\left(\mu\frac{r^2}{2} + \lambda\frac{z^2}{2} + \frac{c}{4}r^4 + \frac{k}{2}r^2z^2 + \frac{b}{4}z^4\right) \tag{33}$$

For a normalization constant N to exist, it is necessary that b and c are negative and $cb - k^2$ is positive in the event that k is positive. We may rewrite Eq. (33) in the original system parameters and variables as

$$p_s(r,z) = N'\, r\exp\left[\frac{2}{(S_1)^2}\left(\mu\frac{r^2}{2} + \frac{c}{4}r^4\right) + \frac{2}{(S_2)^2}\left(\lambda\frac{z^2}{2} + \frac{b}{4}z^4\right) + \frac{1}{2}\left(\frac{d}{(S_2)^2} + \frac{e}{(S_1)^2}\right)r^2z^2\right]$$

$$= N'\, r\exp[\phi_0(r,z)] \tag{34}$$

where the potential flow condition $\delta = 0$ implies that

$$\frac{d}{(S_2)^2} = \frac{e}{(S_1)^2} \tag{35}$$

Bringing Eq. (34) to Cartesian coordinates with

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$$

yields,

$$p_s(x,y,z) = \left[\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|\right]^{-1} p_s(r,\theta,z)$$

$$= N'\exp\{\phi_0[(x^2+y^2)^{1/2}, z]\} \tag{36}$$

It is easy to verify that deterministic fixed points/limit cycles carry over as the extrema of $p_s(x,y,z)$. Since

$$\frac{\partial p_s}{\partial x} = \frac{2}{(S_1)^2}[\mu x + c(x^2+y^2)x] + \left[\frac{d}{(S_2)^2} + \frac{e}{(S_1)^2}\right]xz^2$$

$$= \frac{2}{(S_1)^2}[\mu x + c(x^2+y^2)x + exz^2]p_s$$

using Eq. (35), and similarly,

$$\frac{\partial p_s}{\partial y} = \frac{2}{(S_1)^2}[\mu y + c(x^2+y^2)y + eyz^2]p_s \tag{37a}$$

$$\frac{\partial p_s}{\partial z} = \frac{2}{(S_2)^2}[\lambda z + bz^3 + d(x^2+y^2)z]p_s \tag{37b}$$

the extrema are then defined by

$$0 = \mu x + c(x^2+y^2)x + exz^2 \tag{38a}$$

$$0 = \mu y + c(x^2+y^2)y + eyz^2 \tag{38b}$$

$$0 = \lambda z + bz^3 + dr^2 z \tag{38c}$$

which are precisely the equations defining the bifurcation behavior of the deterministic system. This indicates that the bifurcation behavior of the system is robust provided $d/(S_2)^2 = e/(S_1)^2$. Hence, for the case of coupled Dutch roll/spiral instability, deterministic bifurcation characteristics are preserved if the relative degree of mode interaction and external excitation is the same for each mode.

Near Potential Flow Systems

For the case of small deviation from the potential flow case considered in the last section, it is expedient that a new approach be used in solving the FPE. The idea is a straightforward analogy of integrable and near-integrable systems in Hamiltonian dynamics and is partly inspired by the Hamiltonian formalism employed by Soize.[17] Potential flow systems can be regarded as the ideal case whereby deterministic characteristics are preserved under random excitation and near-potential flow systems as the actual physical situation. The effects of external excitation and mode interaction between the critical modes can then be examined by using regular perturbation theory. The problem can be stated in general terms as follows:

Let $p_0 = N\exp(\phi_0)$ be the steady-state probability density function for the potential flow system governed by the FPE:

$$0 = -\frac{\partial}{\partial x_i}\left[m_{0i}p - \frac{1}{2}Q_{ij}\frac{\partial p}{\partial x_j}\right] \tag{39}$$

Now consider the FPE for the near-potential system:

$$0 = -\frac{\partial}{\partial x_i}\left[(m_{0i} + \epsilon m_{1i})p - \frac{1}{2}(Q_{ij} + \epsilon Q1_{ij})\frac{\partial p}{\partial x_j}\right] \tag{40}$$

Let the solution be given by

$$p = N\exp\left(\phi_0 + \sum_{n=1}\epsilon^n\phi_n\right) = p_0(1 + \epsilon p_1 + \epsilon^2 p_2 + \ldots) \tag{41a}$$

where

$$p_0 = \exp(\phi_0), \quad p_1 = \phi_1, \quad p_2 = \phi_2 + \phi_1^2/2, \ldots \tag{41b}$$

and we define the following operators:

$$L_0[\cdot] = -\frac{\partial}{\partial x_i}\left[m_{0i}(\cdot) - \frac{1}{2}Q_{ij}\frac{\partial(\cdot)}{\partial x_j}\right] \tag{42a}$$

$$L_1[\cdot] = -\frac{\partial}{\partial x_i}\left[m_{1i}(\cdot) - \frac{1}{2}Q_{ij}\frac{\partial(\cdot)}{\partial x_j}\right] \tag{42b}$$

Substituting in the FPE for the near-potential system:

$$(L_0 + \epsilon L_1)\,[p_0(1 + \epsilon_{p1} + \epsilon^2 p_2 + \ldots)] = 0$$

the following system of equations is obtained:

$$L_0[p_0] = 0 \tag{43a}$$

$$L_0[p_1\,p_0] = -L_1[p_0] \tag{43b}$$

$$\ldots$$

$$L_0[p_n\,p_0] = -L_1[p_{n-1}\,p_0] \tag{43c}$$

It should also be noted that since p_0 is the potential flow solution, the operator $L_0[p_n\,p_0]$ can be written as

$$L_0[p_n\,p_0] = -\frac{\partial}{\partial x_i}\left[\left(m_{0i}\,p_0 - \frac{Q_{ij}}{2}\frac{\partial p_0}{\partial x_j}\right)p_n - \frac{Q_{ij}}{2}p_0\frac{\partial p_n}{\partial x_j}\right]$$
$$= \frac{Q_{ij}}{2}\frac{\partial}{\partial x_i}\left[p_0\frac{\partial p_n}{\partial x_j}\right] \tag{44}$$

which is a linear, self-adjoint operator. For the case of small deviation from the potential flow condition, $\delta = \epsilon\delta$, the relevant terms for the rescaled system [Eq. (31)] are explicitly:

$$m_0 = \begin{Bmatrix} \mu r + 1/r + cr^3 + krz^2 \\ \lambda z + bz^3 + kr^2 z \end{Bmatrix}, \qquad m_1 = \begin{Bmatrix} -\delta rz^2 \\ \delta r^2 z \end{Bmatrix}$$

$$Q_{ij} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad Q1_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and the potential flow pdf $[p_0 = N\exp(\phi_0)]$ is given by Eq. (33). Hence, the $\mathcal{O}(\epsilon)$ equation is

$$\frac{\partial}{\partial r}\left(p_0\frac{\partial\phi_1}{\partial r}\right) + \frac{\partial}{\partial z}\left(p_0\frac{\partial\phi_1}{\partial z}\right)$$
$$= \delta\left(\frac{\partial}{\partial r}[-rz^2 p_0] + \frac{\partial}{\partial z}[r^2 z p_0]\right) \tag{45}$$

Although many methods exist for the solution of self-adjoint operator equations, in this problem, mode expansion (Galerkin's method) leads to a simple interpretation of the effects of mode interaction in terms of the moments of the potential pdf. First, define a scalar product

$$\langle u, v\rangle = \int_{-\infty}^{\infty}\int_0^{\infty} u\cdot v\,\mathrm{d}r\,\mathrm{d}z \tag{46}$$

and denote the moments of the potential flow probability density function by

$$M(m,n) = E[r_m z_n] = \int_{-\infty}^{\infty}\int_0^{\infty} r^m z^n p_0\,\mathrm{d}r\,\mathrm{d}z \tag{47}$$

Picking trial functions of the form $r^m z^n$, let

$$\phi_1 = \sum_m\sum_n C_{mn}\,r^m z^n \tag{48}$$

Substituting into the $\mathcal{O}(\epsilon)$ equation, and taking the scalar product with the weight functions denoted by $r^p z^s$,

$$\left\langle r^p z^s,\, L_0\left[p_0\sum_m\sum_n C_{mn}r^m z^n\right]\right\rangle = \langle r^p z^s, -L_1[p_0]\rangle \tag{49}$$

leads to

$$\sum_m\sum_n[mpM(p+m-2,\, s+n) + nsM(m+p,\, s+n-2)]$$
$$\times\, C_{mn} = \delta[sM(p+2,\, s) - pM(p,\, s+2)] \tag{50}$$

It is meaningful to regroup terms in the Itô system representation of Eq. (31):

$$\mathrm{d}r = [(\mu + ez^2)r + cr^3 + 1/r]\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_1(t) \tag{51a}$$

$$\mathrm{d}z = [(\lambda + dr^2)z + bz^3]\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_2(t) \tag{51b}$$

Observe that, with respect to the Dutch roll mode r, the spiral divergence mode z behaves like a linear multiplicative noise source in r through the mode interaction term ez^2r. There is a similar behavior for the divergence mode. Hence, one expects the dominant effects of mode interaction to occur via changes in the unfolding parameters λ and μ. Using a three-mode expansion, $\phi_1 = C_{20}r^2 + C_{11}rz + C_{02}z^2$, Eq. (50) reduces to

$$\begin{bmatrix} 4M(2,0) & 0 & 0 \\ 0 & M(2,0)+M(0,2) & 0 \\ 0 & 0 & 4M(0,2) \end{bmatrix} \begin{Bmatrix} C_{20} \\ C_{11} \\ C_{02} \end{Bmatrix} = \delta \begin{Bmatrix} -2M(2,2) \\ 0 \\ M(2,2) \end{Bmatrix} \tag{52}$$

There, the first-order approximation to the steady-state pdf is

$$p_s(r,z) = N \exp[\phi_0 + \epsilon\phi_1]$$
$$= Nr \exp\left\{\left[\mu - \epsilon\frac{\delta M(2,2)}{2M(2,0)}\right]\frac{r^2}{2} + \left[\lambda + \epsilon\frac{\delta M(2,2)}{2M(0,2)}\right] \times \frac{z^2}{2} + \frac{c}{4}r^4 + \frac{k}{2}r^2z^2 + \frac{b}{4}z^4\right\} \tag{53}$$

and the effective unfolding parameters are defined by

$$\mu' = \mu - \epsilon\frac{\delta M(2,2)}{2M(2,0)}, \qquad \lambda' = \lambda + \epsilon\frac{\delta M(2,2)}{2M(0,2)} \tag{54}$$

The extrema of the pdf Eq. (53) are given by

$$\frac{\partial p_s}{\partial x} = [\mu' x + c(x^2+y^2)x + kxz^2]p_s = 0 \tag{55a}$$

$$\frac{\partial p_s}{\partial y} = [\mu' y + c(x^2+y^2)y + kyz^2]p_s = 0 \tag{55b}$$

$$\frac{\partial p_s}{\partial z} = [\lambda' z + bz^3 + k(x^2+y^2)z]p_s = 0 \tag{55c}$$

These equations take the same form as those defining the fixed points of the deterministic system except that the effective unfolding parameters μ', λ' have to be used instead. It should be noted that the moments $M(2,0)$, $M(0,2)$, and $M(2,2)$ are also functions of μ and λ. Furthermore, since the moments are positive, mode interaction has a beneficial effect on the stability of one mode and is detrimental to the stability of the other. This is evident from Eq. (54) where the correction term is seen to increase the unfolding parameter for one mode but decreases the unfolding parameter for the other. For a positive deviation ($\delta > 0$), the Dutch roll mode will experience the stabilizing effect, whereas a negative deviation ($\delta < 0$) favors the spiral mode.

It is instructive to consider a particular example in detail. Taking the following parameter values for the unscaled deterministic [$\eta(t) = \mathbf{0}$] system of Eqs. (23):

$$b = c = -1, \qquad d = e = -0.5$$

the fixed points of the deterministic system (where defined) are given by

Trivial fixed point:

$$(0,0)$$

Primary bifurcation:

$$(r_p,0) = (\sqrt{\mu},0)$$

$$(0,z_p) = (0,\sqrt{\lambda})$$

Secondary bifurcation:

$$(r_s,z_s) = [\sqrt{(4\mu - 2\lambda)/3},\ \sqrt{(4\lambda - 2\mu)/3}] \tag{56}$$

The following excitation intensities are chosen:

Case 1:

$$S_1 = 1.05, \qquad S_2 = 0.95, \qquad (\delta = -0.025)$$

Case 2:

$$S_1 = 0.95, \qquad S_2 = 1.05, \qquad (\delta = 0.025)$$

The effect of the nonzero deviation δ is apparent in Fig. 1, where the primary bifurcation z_p for the deterministic system is compared with the corresponding extremum of the probability density function for $\mu = -1$. It can be seen that a nonzero deviation from potential flow conditions can either advance or delay the primary bifurcation.

In Fig. 2, the secondary bifurcation is examined for $\lambda = 1$. The z_s component is plotted against μ and compared with the two corresponding stochastic cases. The excitation intensities are the same as earlier. It can be observed that the delay of the primary bifurcation caused by a negative deviation from potential flow conditions (case 1) is also present in the secondary bifurcation. On the other hand, a small change in the excitation intensities resulting in a positive deviation (case 2) has a totally opposite effect. Similar results are obtained for other combinations of system parameters and excitation intensities. This example illustrates the nontrivial effects of external random excitation on the interaction of critical modes and

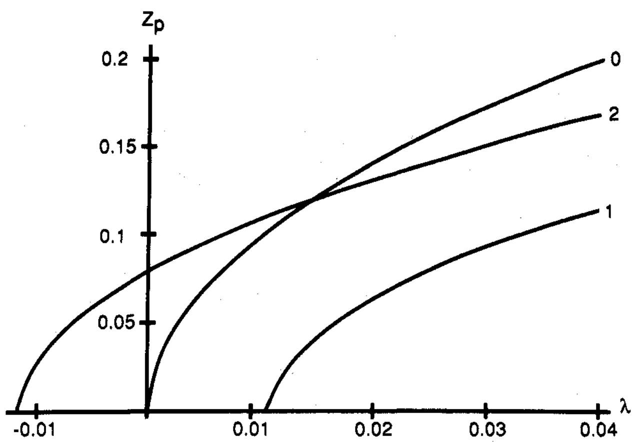

Fig. 1 Primary bifurcation: 0) deterministic; 1) stochastic ($\delta = -0.025$); 2) stochastic ($\delta = 0.025$).

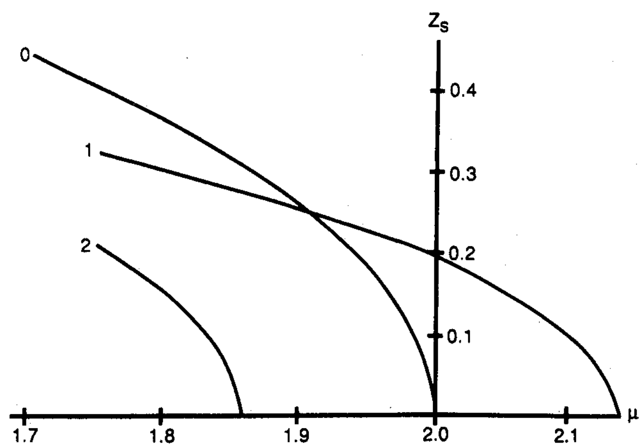

Fig. 2 Secondary bifurcation: 0) deterministic; 1) stochastic ($\delta = -0.025$; 2) stochastic ($\delta = 0.025$).

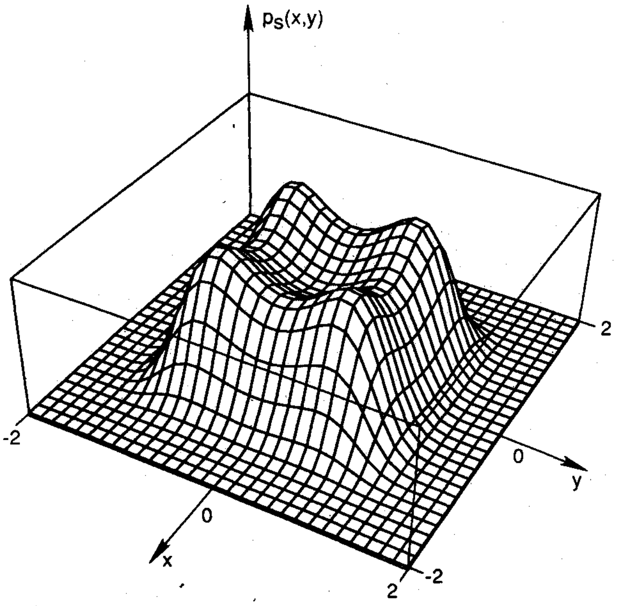

Fig. 3 Probability density function: independent excitation ($k=0$).

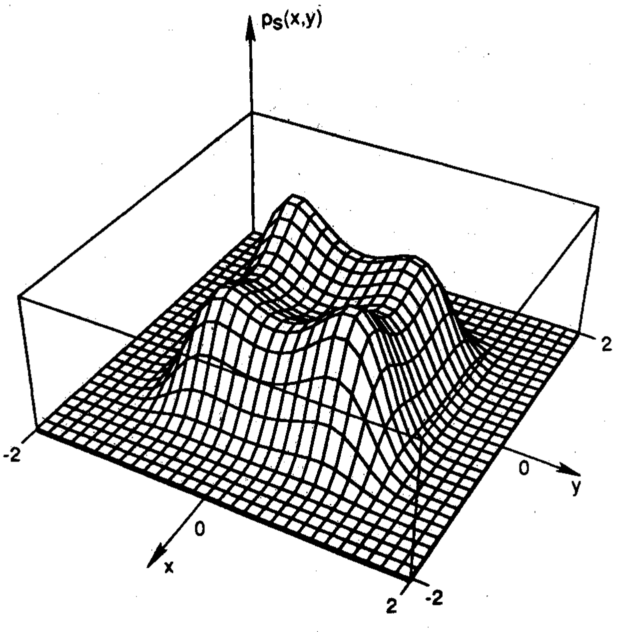

Fig. 4 Probability density function: correlated excitation ($k=0.02$).

demonstrates that the bifurcation behavior of systems with two marginally stable/unstable modes is not robust to external random excitation.

Extension to Other Codimension Two Bifurcation

The perturbation method outlined earlier may also be applied to systems with two pairs of marginally stable/unstable complex eigenvalues or two marginally stable/unstable real eigenvalues. To further emphasize the quasimultiplicative noise effect of critical mode interaction encountered earlier, consider a system with two uncoupled deterministic modes, each capable of a soft loss of stability (simple bifurcation), perturbed by weakly correlated external excitation:

$$\boldsymbol{m}_0=\{\mu x+ax^3,\ \lambda y+by^3\}^T, \qquad \boldsymbol{m}_1=\{0,0\}^T$$

$$Q_{ij}=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}, \qquad Q1_{ij}=\begin{pmatrix}0 & 2k\\ 2k & 0\end{pmatrix} \tag{57}$$

where k is the correlation coefficient of the external excitation on each mode and $a,\ b<0$ for a soft transition. In this case, the mode expansion leads to

$$\sum_m\sum_n [mpM(p+m-2,\ s+n)+nsM(m+p,\ s+n-2)] \times C_{mn}=-2kspM(p-1,\ s-1) \tag{58}$$

the dominant effects of correlated excitation occur through the mode $\phi_1=C_{11}xy$ with

$$C_{11}=\frac{-2k}{M(0,2)+M(2,0)} \tag{59}$$

As can be seen from Figs. 3 and 4, correlated excitation alters the relative values of the extrema of the probability density function with respect to the ideal (potential flow) system ($k=0$), but the unfolding parameters μ and λ are unaffected by the external excitation at the lowest order.

Example: Lateral Flight Dynamics of Rigid Aircraft

Some of the key ideas outlined earlier are now illustrated with a simple example on the lateral dynamics of rigid aircraft. The linear lateral stability derivatives are based on the Piper Cherokee example in McCormick.[18] At a lift coefficient of 1.85, the system in dimensional time is defined by

$$\begin{Bmatrix}\beta'\\ p'\\ r'\\ \phi'\end{Bmatrix}=\begin{bmatrix}-0.0775 & -0.0013 & -0.9946 & 0.3619\\ -3.630 & -2.630 & 1.214 & 0\\ 1.131 & -0.2086 & -0.2477 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}$$

$$\times\begin{Bmatrix}\beta\\ p\\ r\\ \phi\end{Bmatrix}+\begin{Bmatrix}0\\ (l\beta 3)\beta^3\\ (n\beta 3)\beta^3\\ 0\end{Bmatrix}+\begin{Bmatrix}-0.0013\\ -2.630\\ -0.2086\\ 0\end{Bmatrix}P_g(t) \tag{60}$$

The eigenvalues are $-0.0899\pm 1.3543j$, 0.0332, -2.8089. Hence, the Dutch roll mode is marginally stable and the spiral mode is marginally unstable. For simplicity, it is assumed that only the roll and yaw equations exhibit a cubic dependence on the sideslip angle β. The turbulence model is adapted from the material in Roskam,[19] and only the effects of random excitation on the roll rate will be considered. The power spectral density for $P_g(t)$ is given by

$$\Phi(\omega)=\frac{0.006390}{1+0.1842\omega^2} \tag{61}$$

where the intensity is taken to be 21 ft/s and the scale factor as 2500 ft to represent conditions in a thunderstorm. Bringing the system to canonical form and averaging, the cubic coefficients of the reduced system are now linear functions of $l\beta 3$ and $n\beta 3$:

$$c=-(2.3844l\beta 3+2.0933n\beta 3)10^{-2}$$

$$e=-(5.6552l\beta 3+4.9648n\beta 3)10^{-4}$$

$$b=(9.9081l\beta 3+32.091n\beta 3)10^{-5}$$

$$d=(2.5066l\beta 3+8.1185n\beta 3)10^{-2}$$

$$S_1^2=0.04286\int_{-\infty}^{\infty}R_{pp}(\tau)\cos(1.3543\tau)\,\mathrm{d}\tau$$

$$=0.04286\ 2\pi\Phi(1.3543)=1.236\times 10^{-3}$$

$$S_2^2=0.9709\int_{-\infty}^{\infty}R_{pp}(\tau)\,\mathrm{d}\tau$$

$$=0.9709\ 2\pi\Phi(0)=3.899\times 10^{-2}$$

For a soft loss of stability, Eq. (33) states that $b,c<0$, i.e.,

$$n\beta 3<0, \qquad 0.878(-n\beta 3)<l\beta 3<3.228(-n\beta 3) \tag{62}$$

The bifurcation behavior of the deterministic system is preserved provided $\delta=0$ [Eq. (35)]. This leads to

$$l\beta 3=-2.28n\beta 3 \tag{63}$$

which is well within the constraints for a soft transition but occurs only for this special relationship between the nonlinear aerodynamic coefficients and is dependent on the intensity of the external excitation. Furthermore, it is preferable in practice to stabilize the Dutch roll mode at the expense of the spiral mode. This is achieved if $\delta>0$, i.e., the nonlinear coefficients must satisfy

$$l\beta 3>-2.28n\beta 3 \tag{64}$$

Given the actual values of the nonlinear coefficients, Eqs. (62-64) then provide an indication of the dynamics at large angles of attack and sideslip for the level of external excitation assumed.

Conclusions

The major conclusions of this investigations are as follows:

1) The bifurcation behavior of systems with two critical modes is nonrobust in the presence of external random excitation. The deterministic bifurcation behavior is preserved only under special conditions. Otherwise, random external excitation can modify the unfolding parameters through the system nonlinearities.

2) It is not the actual intensity of the excitation that is important but rather the relative degree of mode interaction and intensity of external excitation in each mode that determines the bifurcation behavior.

3) For the case of marginally stable/unstable Dutch roll/spiral modes, external random excitation has an opposite effect on the two unfolding parameters. In practice, this indicates that one mode is stablized by external random excitation only at the expense of the other mode.

Finally it should be noted that the ideas outlined here are not restricted to problems in flight dynamics. In aeroelasticity, a fairly similar situation occurs. The governing partial differential equations in aeroelasticity can be reduced to a system of ordinary differential equations by means of modal expansion.[20] Flutter instability then corresponds to a Hopf bifurcation and divergence instability to a simple bifurcation. Similar conclusions with regard to the effects of external random excitation can be made.

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